The Optimality of an Easily Implementable Feedback Control System: An Inverse Problem in Optimal Control Theory

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In most control applications in the chemical process industries it is not realistic to attempt to define a unique mathematical statement of the control objective, for many criteria will satisfy the physical requirement of the rapid elimination of errors in the product stream as the result of an upset. The strong dependence of the structure of an optimal control system on the choice of objective then makes optimal control theory irrelevant in such situations, since the control engineer has no assurance that a complicated controller is a necessity of the process, rather than a consequence of an unfortunate choice of objective. In this paper an inverse problem is considered, in which an easily implementable feedback control system is first chosen and then is shown to be optimal for a physically meaningful objective in a large class of systems.

The greatest number of control applications which arise in the chemical process industries are such that no precise cost function can be formulated to define the control objective, although the qualitative requirements of an effective controller are clear. In the simplest cases the requirement is that fluctuations in certain elements of the product stream be kept small following an upset, and a large number of mathematical statements will satisfy this physical requirement. It might be stated, for example, that the system should be returned to steady state operating conditions in the minimum time, or perhaps that an integral of the squares or absolute values of the deviations from steady state be made as small as possible over the operating period. If a model of the plant is available in the form of difference or differential equations, then the attainment of any of these objectives may be formulated as a problem in optimal control, with the solution obtained with such techniques as dynamic programming (*1*) or the Pontryagin minimum principle (*4, 17*).

The serious drawback in the use of the modern mathematics of control theory from the point of view of the process industries, however, stems from the fact that the structure, as well as the detailed design of the optimal control system, depends critically on the particular form of the objective which has been chosen. In a linear plant, for example, the control which minimizes the time of return to steady state is of relay, or bang-bang type, while the control which minimizes the integral of the squared deviations in state and control is proportional to the deviations. Equally meaningful objectives lead to still other feedback controls, some of which might be quite difficult to implement, and for complicated systems it will be necessary to solve the optimal control problem by gradient types of relaxation methods (*4, 9*), which generally result in a programmed, rather than feedback, control. It would seem, then, that the usual optimal control problem is largely irrelevant in such situations, for the engineer has no assurance that the resulting optimal solution obtained from a somewhat arbitrary choice of objective is not significantly more complicated and expensive than the result which might be obtained from a physically equivalent, but less obvious choice of objective.

Under certain circumstances this difficulty can be overcome by consideration of the *inverse problem* of optimal control. Here a feedback control system is first specified, and then it is determined under what conditions this control is the solution of an optimal control problem. Several authors (*7, 14, 18*) have considered aspects of this problem, but the significance of the results for the process engineer is limited in each case because of the fact that the form of the objective has been specified, with only parameters investigated. In this paper a different point of view is adopted, and only the form of the plant and the feedback control system are specified, the latter being an easily implementable relay controller outside some region of the steady state, with optional modifications to eliminate chatter near the steady state. It is then shown that, subject to certain stability conditions on the uncontrolled plant, the simple relay controller does optimize a physically meaningful control objective. Another paper (*13*) will consider the optimality of the frequently used three-mode control.

THE RELAY CONTROLLER

For the first part of the discussion only, it will be assumed that the plant is linear and stationary, so that deviations from the steady state design conditions may be represented by the N ordinary differential equations:

$$\dot{x}_i = \sum_{j=1}^{N} A_{ij} x_j + \sum_{k=1}^{M} B_{ik} u_k, \quad i = 1, 2, \ldots, N \qquad (1)$$

Here $x_1, x_2, \ldots, x_N$ are the deviations in the N variables required to define the state of the system and $u_1, u_2, \ldots, u_M$ are the adjustments in the M control variables. The control action is assumed to be bounded, and it will be further assumed that the bounds are symmetric in the positive and negative directions. In that case the elements of the matrix **B** may be defined so that the control is normalized, and there is no loss of generality in assuming that

$$|u_k| \leq 1, \quad k = 1, 2, \ldots, M \qquad (2)$$

In all that follows the assumption of symmetric bounds leads to simplification in certain expressions but the overall conclusions in no way depend upon this simplification.

The usual optimal control problem calls for choosing the control vector $\mathbf{u}(t)$ [made up of components $u_1(t)$, $u_2(t)$, ..., $u_M(t)$] in order to minimize some overall measure of system performance in the form

$$\mathcal{E} = \int_0^\theta F(\mathbf{x}, \mathbf{u}, t)\, dt \tag{3}$$

where θ is the total control interval and $\mathbf{x}$ is the vector with components $x_1(t)$, $x_2(t)$, ..., $x_N(t)$. In most process applications the "cost" of control action is negligible, and fluctuations are as costly at one time as at any other. Thus in a physically meaningful formulation the function F will generally be independent of $\mathbf{u}$ and t, and

$$\mathcal{E} = \int_0^\theta F(\mathbf{x}) dt \tag{4}$$

In order to force the system to return to steady state, it is necessary that $F(\mathbf{x})$ be positive definite in any dependence on $\mathbf{x}$; that is

$$F(\mathbf{x}) > F(\mathbf{o}) \geqq o, \quad \mathbf{x} \neq \mathbf{o} \tag{5}$$

It may be that $F(\mathbf{o}) > o$, in which case a premium on total control time is included in the objective.

Several authors (*2, 3, 6, 8, 10, 15, 16*), and most recently Paradis and Perlmutter (*15, 16*), have recognized the difficulty in solving and implementing feedback control systems which optimize the overall objective of Equation (4) and they have focused attention on instantaneously optimal policies. For example, the positive definite quadratic form

$$E = \frac{1}{2} \sum_{i,j=1}^{N} x_i\, Q_{ij}\, x_j \tag{6}$$

is a measure of the instantaneous deviation from steady state. An instantaneously optimal policy would then be one which drives E to zero as fast as possible; that is, one which minimizes

$$\dot{E} = \sum_{i,j,k=1}^{N} x_i\, Q_{ij}\, A_{jk}\, x_k + \sum_{i,j=1}^{N} \sum_{k=1}^{M} x_i\, Q_{ij}\, B_{jk}\, u_k \tag{7}$$

at every instant of time. (Note that there is no loss of generality in taking $Q_{ij} = Q_{ji}$.) This policy cannot take into account the effect on the future of present action. The control action which minimizes Equation (7), since u appears linearly, is clearly

$$u_k = -\,\mathrm{sgn}\left[\sum_{i,j=1}^{N} x_i\, Q_{ij}\, B_{jk}\right], \quad k = 1, 2, \ldots, M \tag{8}$$

where the sgn (signum) function is defined by

$$\mathrm{sgn}\,(y) = \begin{cases} +\,1, & y > 0 \\ -\,1, & y < 0 \end{cases} \tag{9}$$

This is a relay feedback controller with a linear switching law, requiring only a minimum of on-line equipment for implementation. The behavior of such control systems is well known (*5*), and effective control has been reported on both simulated and real processes (*2, 3, 15, 16*).

Because of the effectiveness and ease of implementation of the controller defined by Equation (8), it is of interest to ask the following inverse problem in optimal control:

Do conditions exist under which a relay controller with a linear switching surface corresponds to a physically meaningful optimum for a linear stationary plant? If such conditions can be found then, when a precise mathematical cost function cannot be defined, the simple controller designed by Equation (8) will be known with full confidence to be satisfactory in an overall, as well as an instantaneous, sense, and the more complicated control laws usually obtained from optimal control theory will be unnecessary. It should be noted here that relay controls generally give unsatisfactory performance near the steady state, and an important part of the inverse question is when, if at all, should the relay control be abandoned for some other policy.

THE MINIMUM PRINCIPLE

The primary mathematical tool which will be required for the solution of the inverse problem is the minimum principle (*4, 17*), which, for present purposes, is loosely as follows:

The state of the system is described by the N ordinary differential equations

$$\dot{x}_i = g_i\,(\mathbf{x}, u), \quad i = 1, 2, \ldots, N \tag{10}$$

and it is desired to choose the control vector function $\mathbf{u}(t)$ to move the system to a given surface in $\mathbf{x}$ space,

$$S(\mathbf{x}) = 0 \tag{11}$$

while minimizing an overall objective

$$\mathcal{E} = \int_0^\theta F(\mathbf{x}, \mathbf{u}) dt \tag{12}$$

The variational Hamiltonian is defined as

$$H = F(\mathbf{x}, \mathbf{u}) + \sum_{i=1}^{N} \lambda_i\, g_i(\mathbf{x}, \mathbf{u}) \tag{13}$$

where the Lagrange multipliers λ_i satisfy the equations

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial F}{\partial x_i} - \sum_{j=1}^{N} \lambda_j \frac{\partial g_j}{\partial x_i}, \; i = 1, 2, \ldots, N \tag{14}$$

with boundary conditions

$$\lambda_i(\theta) = \lambda \frac{\partial S}{\partial x_i}, \quad i = 1, 2, \ldots, N \tag{15}$$

with λ a constant. [That is, $\underset{\sim}{\lambda}(\theta)$ is orthogonal to the surface $S = 0$.] *It is necessary that the control function* $\mathbf{u}(t)$ *which minimizes the objective Equation* (12) *minimize the Hamiltonian at every instant in time. Furthermore, the Hamiltonian is a constant along the optimal path, with the value zero if the total control interval is unspecified.*

THE OVERALL OBJECTIVE

The Hamiltonian for the linear stationary plant described by Equation (1) with the objective described by Equation (4) is

$$H = F(\mathbf{x}) + \sum_{i,j=1}^{N} \lambda_i\, A_{ij}\, x_j + \sum_{i=1}^{N} \sum_{k=1}^{M} \lambda_i\, B_{ik}\, u_k \tag{16}$$

Since $\mathbf{u}$ enters the Hamiltonian linearly the control which minimizes H has the form

$$u_k = -\,\mathrm{sgn}\left[\sum_{i=1}^{N} \lambda_i\, B_{ik}\right], \quad k = 1, 2, \ldots, M \tag{17}$$

Comparison with Equation (8) indicates that the relay control with linear switching can be optimal only if the functions $\sum_{i,j=1}^{N} x_i\, Q_{ij}\, B_{jk}$ and $\sum_{j=1}^{N} \lambda_j B_{jk}$ always have the same algebraic sign; that is

$$\sum_{j=1}^{N} \lambda_j B_{jk} = a_k(t) \sum_{i,j=1}^{N} x_i Q_{ij} B_{jk}, \quad a_k(t) > 0, \quad k = 1, 2, \ldots, M \tag{18}$$

or, equivalently

$$\lambda_j(t) = a_k(t) \sum_{i=1}^{N} x_i(t) Q_{ij} + m_j(t), j = 1, 2, \ldots, N \tag{19}$$

with

$$\sum_{j=1}^{N} m_j(t) B_{jk} = 0, \quad k = 1, 2, \ldots, M \tag{20}$$

Since $a_k(t)$ must clearly be the same for all k in Equation (19), the subscript may be dropped, and the condition that the controllers be the same is

$$\lambda_j(t) = a(t) \sum_{i=1}^{N} x_i(t) Q_{ij} + m_j(t), j = 1, 2, \ldots, N \tag{21}$$

The multipliers $\lambda_i(t)$ must satisfy Equation (14), which becomes

$$\dot{\lambda}_i = -\frac{\partial F}{\partial x_i} - \sum_{j=1}^{N} \lambda_j A_{ji}, \quad i = 1, 2, \ldots, N \tag{22}$$

and substituting Equation (21)

$$\dot{\lambda}_i = -\frac{\partial F}{\partial x_i} - a(t) \sum_{j,k=1}^{N} x_k Q_{kj} A_{ji} - \sum_{j=1}^{N} m_j A_{ji}, \quad i = 1, 2, \ldots, N \tag{23}$$

But differentiating Equation (21) and using Equation (8), we get

$$\dot{\lambda}_i = \dot{a}(t) \sum_{j=1}^{N} Q_{ij} x_j + a(t) \sum_{j,k=1}^{N} Q_{ij} A_{jk} x_k - a(t) \sum_{j=1}^{N} \sum_{k=1}^{M} Q_{ij} B_{jk} \operatorname{sgn}\left[\sum_{r,q=1}^{N} x_r Q_{rq} B_{qk} \right] + \dot{m}_i(t), \quad i = 1, 2, \ldots, N \tag{24}$$

Since the right-hand sides of Equations (23) and (24) must represent the same quantity, they may be equated, which leads to a set of partial differential equations for the integrand $F(\mathbf{x})$:

$$\frac{\partial F}{\partial x_i} = -a(t) \sum_{j,k=1}^{N} x_k (Q_{kj} A_{ji} + Q_{ij} A_{jk}) - \dot{a}(t) \sum_{j=1}^{N} Q_{ij} x_j + a(t) \sum_{j=1}^{N} \sum_{k=1}^{N} B_{jk} \operatorname{sgn}\left[\sum_{r,q=1}^{N} x_r Q_{rq} B_{qk} \right] - \sum_{j=1}^{N} m_j(t) A_{ji} - \dot{m}_i(t), \quad i = 1, 2, \ldots, N \tag{25}$$

In order for F to be free of explicit dependence on time, it is necessary that $a(t)$ and $m_i(t)$ be constants, so that the $\dot{a}(t)$ and $\dot{m}_i(t)$ terms vanish. [The alternative, that

$$\sum_{j=1}^{N} m_j A_{ji} + \dot{m}_i = 0$$

is ruled out in general by Equation (20)]. Since a multiplicative constant is immaterial $a(t)$ may then be set without loss of generality to unity, and Equation (25) may be integrated to give

$$F(\mathbf{x}) = -\frac{1}{2} \sum_{i,j,k=1}^{N} x_i (Q_{kj} A_{ji} + Q_{ij} A_{jk}) x_k - \sum_{i,j=1}^{N} m_i A_{ij} x_j + \sum_{k=1}^{M} \left| \sum_{i,j=1}^{N} x_i Q_{ij} B_{jk} \right| + \text{constant} \tag{26}$$

Equation (5), the requirement that F take on its minimum at $\mathbf{x} = \mathbf{0}$, can be satisfied only if

$$\sum_{i=1}^{N} m_i A_{ij} = 0, \quad j = 1, 2, \ldots, N \tag{27}$$

which, together with Equation (15), implies in general that

$$m_i = 0, \quad i = 1, 2, \ldots, N \tag{28}$$

Since the sum of absolute values in Equation (26) is only positive semidefinite (that is, it may be zero for nonzero $\mathbf{x}$), F will be positive definite if and only if the quadratic form

$$\sum_{i,j,k=1}^{N} x_i (Q_{kj} A_{ji} + Q_{ij} A_{jk}) x_k$$

is negative definite. Finally, if the total control interval θ is unspecified, then the requirement that the Hamiltonian vanish implies, after substitution of Equations (17), (21), (26), and (27) into Equation (16), that the constant in Equation (26) is zero. If θ is specified then the value of the constant is irrelevant to the optimization and may be disregarded. Thus the minimum principle gives as a candidate for the integrand of an overall objective for which Equation (8) is the optimal control the function

$$F(\mathbf{x}) = -\frac{1}{2} \sum_{i,j,k=1}^{N} x_i (Q_{ij} A_{jk} + Q_{kj} A_{ji}) x_k + \sum_{k=1}^{M} \left| \sum_{i,j=1}^{N} x_i Q_{ij} B_{jk} \right| \tag{29}$$

where the matrix $\sum_{k=1}^{N} (Q_{ij} A_{jk} + Q_{kj} A_{ji})$ is negative definite; that is, all quadratic forms obtained from the matrix are negative definite.

STABILITY CONSIDERATIONS

In order to interpret the result obtained above, it is necessary to recall some elementary results from the theory of Liapunov stability of ordinary differential equations (*8, 11*) of the form

$$\dot{x}_i = G_i(\mathbf{x}), \quad i = 1, 2, \ldots, N \tag{30}$$

with an equilibrium point at the origin; that is, the steady state satisfies the equations

$$G_i(\mathbf{0}) = 0, \quad i = 1, 2, \ldots, N \tag{31}$$

Consider any function $V(\mathbf{x})$ which is positive definite in a region of the origin ($V(\mathbf{x}) > 0$, $\mathbf{x} \neq \mathbf{0}$; $V(\mathbf{0}) = 0$) and which is continuous, together with its first partial derivatives.

1. If $\dot{V}(\mathbf{x}) \leq 0$ in a region of the origin ($\dot{V}$ negative semidefinite), then $V(\mathbf{x})$ is called a Liapunov function and the origin is stable.
2. If $\dot{V}(\mathbf{x})$ is negative definite then the origin is asymptotically stable, which means that as $t \to \infty$ all trajectories approach $\mathbf{x} = \mathbf{0}$.

The uncontrolled plant described by Equation (1) is

$$\dot{x}_i = \sum_{j=1}^{N} A_{ij} x_j, \quad i = 1, 2, \ldots, N \tag{32}$$

and for this system

$$\dot{E} = \frac{1}{2} \frac{d}{dt} \sum_{i,j=1}^{N} x_i Q_{ij} x_j$$

$$= \frac{1}{2} \sum_{i,j,k=1}^{N} x_i (Q_{ij} A_{jk} + Q_{kj} A_{ji}) x_k \tag{33}$$

which must be negative definite in order for $F(\mathbf{x})$ to define a physically meaningful objective. Thus it is necessary that the positive definite quadratic form E defined by Equation (6) be a Liapunov function for the uncontrolled system, and that the uncontrolled system be asymptotically stable, in order for the relay controller with linear switching to be optimal in an overall sense. Under these conditions the function $\dot{E}$ defined by Equations (7) and (8) is always negative definite, and hence E is a Liapunov function for the controlled system as well, and the controlled plant is assured to be stable.

This result relates the present work to earlier studies of controller design by rapid reduction of a Liapunov function (*3, 8, 10, 19*), and establishes the optimality of such a procedure for at least the limited case of a quadratic Liapunov function. It is interesting to note that Kalman (*7*) obtained a similar relation between system stability and optimality for proportional control and quadratic objective.

OPTIMALITY AND UNIQUENESS

The control function defined by Equation (8) has now been used to derive a criterion of optimization for the linear plant:

$$\mathcal{E} = \int_0^\theta \left\{ \frac{1}{2} \sum_{j,k=1}^{N} x_i C_{ij} x_j + \sum_{k=1}^{M} \left| \sum_{i,j=1}^{N} x_i Q_{ij} B_{jk} \right| \right\} dt \tag{34}$$

if and only if the uncontrolled plant is asymptotically stable, where

$$C_{ij} = - \sum_{k=1}^{N} (Q_{ik} A_{kj} + Q_{jk} A_{ki}); \quad i, j = 1, 2, \ldots, N \tag{35}$$

is positive definite. The asymptotic stability of the uncontrolled system is sufficient to ensure the existence of a positive definite matrix Q_{ij}, as required for the E function, for an arbitrary positive definite C_{ij} (*11*), although the converse is not true. The control system is easily implementable and the criterion is exceedingly reasonable in terms of the physical objective of reducing and keeping small the deviations from steady state. Indeed, quadratic forms are often used for the function $F(\mathbf{x})$, and the addition of an unimportant positive semidefinite term simply shapes the trajectories somewhat differently, while allowing a simplification of the control policy. It remains to be shown, however, that the control policy which was used to obtain the objective Equation (34) is in fact the control which minimizes that objective. This problem arises because the minimum principle is not a sufficient condition for optimality and there may be multiple solutions.

The control law expressed by Equations (8) and (17) is based on the assumption that the linear switching function can never vanish for a finite-time interval, for in such a case the signum function is undefined and the optimum may be given by an intermediate "singular" control policy. It is first necessary to show, then, that for the minimization of the objective Equation (34), the function

$$\sum_{i,j=1}^{N} x_i Q_{ij} B_{jk}$$

cannot vanish for any finite interval under optimal control. It is particularly important to consider this possibility for the common case of a single control variable, for should the switching function vanish the objective would then have only a quadratic form in the integrand, and the theory for that case is well known (*20*).

It is convenient to demonstrate that singular solutions cannot occur by means of the example of a second-order system with a single control variable. Without loss of generality such a system may be expressed as

$$\dot{x}_1 = x_2 \tag{36a}$$

$$\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + bu \tag{36b}$$

so that the switching surface defined by Equation (8) is

$$Q_{12} x_1 + Q_{22} x_2 = 0 \tag{37}$$

If this is differentiated with respect to time and Equation (29) substituted, the resulting control is found to be

$$u = - x_1 \left(A_{21} - \frac{Q_{12}^2}{Q_{22}^2} \right) - A_{22} x_2 \tag{38}$$

On the other hand, the vanishing of Equation (37) means that the objective now has an integrand $\frac{1}{2}(C_{11} x_1^2 + 2 C_{12} x_1 x_2 + C_{22} x_2^2)$, and the condition for singular control with this objective is (*20*)

$$u = - x_1 \left(A_{21} - \frac{C_{11}}{C_{22}} \right) - A_{22} x_2 \tag{39}$$

and

$$\sqrt{C_{11}} x_1 + \sqrt{C_{22}} x_2 = 0 \tag{40}$$

Thus, singular control is possible and, in fact, optimal, if and only if

$$\frac{C_{11}}{C_{22}} = \frac{Q_{12}^2}{Q_{22}^2} \tag{41}$$

Together with Equation (35) this yields only discrete values for the ratio C_{11}/C_{22}, and since the $\mathbf{C}$ matrix is at the disposal of the designer only infinitesimal changes are needed to avoid the possibility of intermediate control. Indeed, if $\mathbf{Q}$ is taken to be diagonal then $\mathbf{C}$ will not be definite when Equation (41) is satisfied.

The generalization to higher dimensions is straightforward and yields the same result. The generalization to more than one control variable is not so easy, but it is reasonable to presume that the result established rigorously for a single control variable will also hold in the more complicated case, since the conditions for singularity will be even more stringent.

The assumptions concerning the plant and objective equations are sufficient to ensure the existence of a solution to the optimal control problem (*12*), and that solution must satisfy the minimum principle. Thus, if a unique solution to the minimum principle equations can be established, that solution must be optimal. It is here that the boundary conditions for the multipliers are used.

It will be assumed that the control effort is to terminate upon some manifold

$$\sum_{i,j=1}^{N} x_i(\theta) Q_{ij} x_j(\theta) - \text{constant} = 0 \tag{42}$$

where the final time θ is unspecified. Within this ellipsoidal surface the control is to be of some form other than relay in order to avoid chattering. The boundary condition Equation (15) is then

$$\lambda_i(\theta) = \lambda \sum_{j=1}^{N} Q_{ij}\ x_j(\theta), \quad i = 1, 2, \ldots, N \qquad (43)$$

where λ is some constant, and substitution into the Hamiltonian, Equation (13), immediately establishes that $\lambda = 1$ for $H = 0$. Hence, for any $\mathbf{x}(\theta)$ a complete set of conditions is available for the coupled Equations (1), (17), and (22). Since the switching function can never vanish for a finite interval, the control is always given by Equations (17), so that the solution to these equations is unique. Thus the solution obtained by the use of Equation (19) must be the optimal solution and the relay controller defined by Equation (8) the optimum for Equation (34) while bringing the system to any terminal manifold of the form Equation (42).

NONLINEAR PROCESSES

A large number of processes, including chemical reaction systems, may be described by equations of the form

$$\dot{x}_i = f_i(\mathbf{x}) + \sum_{k=1}^{M} B_{ik}(\mathbf{x})\ u_k, \quad i = 1, 2, \ldots, N \quad (44)$$

The simple relay control obtained by instantaneous minimization of the quadratic form Equation (6) is still described by Equation (8):

$$u_k = -\operatorname{sgn}\left[\sum_{i,j=1}^{N} x_i\ Q_{ij}\ B_{jk}(\mathbf{x}),\right] k = 1, 2, \ldots, M \qquad (45)$$

although the switching functions are no longer planar. This control has been used successfully by Paradis and Perlmutter (*15, 16*) on several nonlinear reactor problems, including a distributed system.

The Hamiltonian for minimization of an overall objective is

$$H = F + \sum_{i=1}^{N} \lambda_i\ f_i + \sum_{i=1}^{N} \sum_{k=1}^{M} \lambda_i\ B_{ik}\ u_k \qquad (46)$$

where

$$\dot{\lambda}_i = -\frac{\partial F}{\partial x_i} - \sum_{j=1}^{N} \lambda_j \frac{\partial f_j}{\partial x_i} - \sum_{j=1}^{N} \sum_{k=1}^{M} \lambda_j \frac{\partial B_{jk}}{\partial x_i} u_k, \quad i = 1, 2, \ldots, N \qquad (47)$$

and the minimum is obtained from

$$u_k = -\operatorname{sgn}\left[\sum_{i=1}^{N} \lambda_i\ B_{ik}\right], \quad k = 1, 2, \ldots, M \quad (48)$$

By identifying λ_i with $\sum_{j=1}^{N} Q_{ij}\ x_j$ and repeating the procedure used for the linear plant, it follows that F is described by the partial differential equations

$$\frac{\partial F}{\partial x_i} = -\sum_{k=1}^{N} Q_{ik}\ f_k - \sum_{j,k=1}^{N} x_j\ Q_{jk} \frac{\partial f_k}{\partial x_i} + \sum_{r=1}^{M}\left[\sum_{j=1}^{N} Q_{ij}\ B_{jr} + \sum_{j,k=1}^{N} x_j\ Q_{jk} \frac{\partial B_{kr}}{\partial x_i}\right] \operatorname{sgn} \left[\sum_{p,q=1}^{N} x_p\ Q_{pq}\ B_{qr}\right], i = 1, 2, \ldots, N \qquad (49)$$

The solution, to within a constant, is

$$F(\mathbf{x}) = -\sum_{i,j=1}^{N} x_i\ Q_{ij}\ f_j + \sum_{k=1}^{M} \left| \sum_{i,j=1}^{N} x_i\ Q_{ij}\ B_{jk} \right| \qquad (50)$$

This result defines a physically meaningful objective only if

$$\sum_{i,j=1}^{N} x_i\ Q_{ij}\ f_j\ (\mathbf{x}) < 0, \quad \mathbf{x} \neq \mathbf{0} \qquad (51)$$

or the quadratic form E in Equation (6) is a Liapunov function for the uncontrolled system. In that case it is a Liapunov function for the controlled plant and the controller is stable. Thus, with the exception of the curvature of the switching function, the essential result for the linear plant is valid for the nonlinear plant as well. For the nonlinear system the requirement that E be a Liapunov function is more restrictive than simply requiring asymptotic stability of the uncontrolled system, however. The existence of a quadratic form as a Liapunov function is ensured only in the region in which linearization is valid, and large regions of asymptotic stability may exist in which no quadratic Liapunov function can be found.

CONCLUDING REMARKS

The primary result of this work is to establish that for most process control problems where the plant is inherently stable and may be approximated as linear and stationary, it is unnecessary to solve an optimal control problem in the usual sense, because controller design based on instantaneous minimization of a quadratic Liapunov function will give results which are optimal in a physically meaningful overall sense. For this to be true, however, the design should be based, not on the quadratic form E defined by Equation (6), as in the work of Paradis and Perlmutter (*15, 16*), but on the quadratic defined by the $\mathbf{C}$ matrix of Equation (35). The question of the value of E at which to change from relay to intermediate—say, proportional—control is governed by practical considerations in avoiding chatter. It is clear, however, that optimality requires that the change be based only on position in the state space (that is, a feedback criterion), rather than an arbitrary open-loop criterion such as the number of switches, as suggested previously (*15*).

For use in nonlinear systems the requirement of a quadratic Liapunov function means that in a region bounded away from the origin the control may be stable, but not optimal in an overall sense, and an alternative proof of stability is required. Such control will still be highly satisfactory and, indeed, optimal over most of the control period. Most of the examples of Paradis and Perlmutter (*15, 16*) fall into this category, for in most cases the E function can be seen to increase during the early periods of operation and thus it cannot be a Liapunov function. In some of these situations the control policy is unchanged by changes in $\mathbf{Q}$ which do make E a Liapunov function.

Finally, it is useful to note that the interest in inverse problems far transcends the immediate area of control theory. The ultimate inverse problem is that of using the observed behavior of physical systems to deduce the underlying variational principles and constitutive relations which govern the system response. The control problem considered here is simply an early contribution to understand the techniques which will be needed for the more far-reaching problems.

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Linear, Steady, Two-Dimensional Flows of Viscoelastic Liquids

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The general form of the equations describing a steady, two-dimensional flow of an incompressible liquid is reduced to a form containing only two parameters. The histories of stress and of deformation of a material element are written explicitly. A second-order, slow-flow approximation and a Maxwell type of constitutive equation are used to infer properties of the rheological behavior of viscoelastic liquids in such flows.

Engineering analysis of viscoelastic fluid mechanics requires the understanding of as large a variety of flow patterns as possible, so that flows of actual interest can be approximated satisfactorily with idealized flows which are rigorously tractable yet reasonably similar to the real ones. Furthermore, the rheological characterization of real fluids would be more reliable and complete if experiments could be made under conditions which are not restricted to only the classical viscometric flow patterns.

A substantial fraction of the literature concerning viscoelastic fluid mechanics is devoted to the analysis of viscometric flows (see, for example, reference *6*). A few different flow patterns have been considered in the literature: elongational flow (*4*), pure shear (*11*), and fourth-order flow (*2*).

If attention is limited to steady, two-dimensional flow patterns, only viscometric flow and pure shear have been considered; Astarita (*2*) has concisely considered a more complex flow pattern which results from a superposition of viscometric flow and pure shear.

In this paper, the general problem of steady, two-dimensional flow is analyzed in considerable detail. A complete description of the deformation history and of the stress history, in the sense of Oldroyd (*11*), of a material element is given explicitly. The two histories change continuously, at constant total shear rate, when the relative importance of the two independent components of the shear rate changes. Thus, if experiments are made for a real fluid, information can be obtained over a variety of different rheological histories. As suggested by Oldroyd (*12*), a sort of bookkeeping of such information may be the basis on which more general forms of the constitutive equation may be inferred.

BASIC KINEMATICS

The most general possible steady, two-dimensional flow pattern of an incompressible fluid can be described, in a locally Cartesian frame z'^i, by the following equations for